Apr 5: Group actions and the structure of roots

Today's outline

- Review group actions
- Structure of roots of polynomials
- Symmetric functions
- (Wed) Lagrange's soln to quartic

§1. Group actions

Let G be a group.

Defn A <u>group action of G on a set X</u> is a law that defines $g\cdot x \in X$ for any $g\in G$, $x\in X$

$\left(G\times X \longrightarrow X \text{ map} \right)$

$(g,x) \longmapsto gx$

satisfying

(1) $(gh)x = g\cdot(hx)$

(2) $e\cdot x = x$ (where $e\in G$ is the identity)

Ex 1 G acts on G via multiplication

$g\in G,\ x\in G \quad g\cdot x = \underbrace{gx}_{\text{mult. in } G}$

Ex 2 G act G via conjugation

$g\in G,\ x\in G \quad \underbrace{g\cdot x}_{\text{action}} := \underbrace{gxg^{-1}}_{\text{mult in } G}$

Defn Given G acting on X and an element $x\in X$, we define

- the <u>orbit</u> $Gx = \{g\cdot x \mid g\in G\}$
- the <u>stabilizer</u>

$$G_x = \{g\in G \mid g\cdot x = x\}$$

In Ex 2, if $x\in G$

orbit $Gx = \{gxg^{-1} \mid g\in G\}$ "conjugacy class"

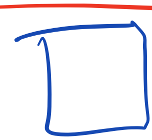
$G_x = \{g\in G \mid \underbrace{gxg^{-1} = x}_{gx = xg}\}$

"centralizer of x"

Defn Given G acting on X and an element $x \in X$, we define

- the **orbit** $Gx = \{ g \cdot x \mid g \in G \}$
- the **stabilizer** $G_x = \{ g \in G \mid g \cdot x = x \}$

Ex 3

G = group of symmetries of the square  $\subset \mathbb{R}^2$

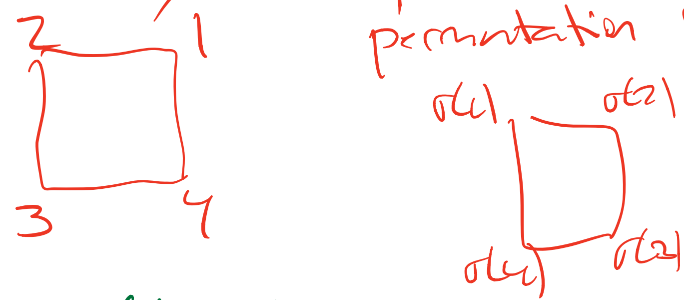
here: symmetry is an invertible 2×2 real matrix (i.e. linear transform of $\mathbb{R}^2$) that preserves the square

- rotation 90° clockwise r
- reflection across x-axis s

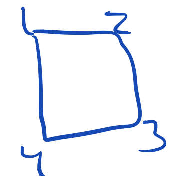
$$G = \langle r, s \mid r^4 = s^2 = id, \; srs^{-1} = r^3 \rangle = D_4$$

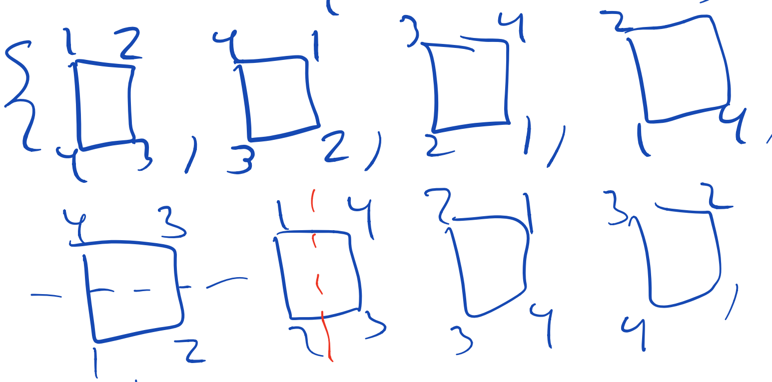
8

Let X = labellings of the vertices of the square

Ex: 

24 elements

orbit of $x =$ 

$Gx = \{$  $\}$

4 1
3 2

$$G_x = \{ e \}$$

FACT

Group action fact: G acting on X

If $x \in X$, then

$$(\# Gx) \cdot (\# G_x) = \# G$$

Reason $G \xrightarrow{f} X$

$g \longmapsto gx$

$\mathrm{im}(f) = Gx$ orbit

$\{g \in G \mid f(g) = x\} = G_x$ stabilizer

$\rightsquigarrow G/G_x \underset{\text{bij}}{\cong} Gx$

$$\frac{(\# G)}{(\# G_x)} = \# Gx$$

$x^2+y^2 \in k[x,y]$ symmetric

$x^2+y^2 \in k[x,y,z]$ not symm

$x^2+y^2+z^2 \in k[x,y,z]$ symm

symmetric

Ex 4 k field

$k[x_1,\dots,x_n]$ polynomial ring

There is an action of S_n on $k[x_1,\dots,x_n]$ defined by:

given $\sigma \in S_n$ & $f \in k[x_1,\dots,x_n]$

$$(\sigma \cdot f)(x_1,\dots,x_n) := f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

Ex: S_2 acting $k[x,y]$

$(1\,2)\cdot(x+y^2) = y+x^2$ not symmetric

$(1\,2)\cdot(xy) = xy$

Defn A polynomial $f \in k[x_1,\dots,x_n]$ is symmetric if $\forall \sigma \in S_n$ $\sigma\cdot f = f$

Equivalently, stabilizer $G_f = S_n$

$\iff$ orbit $Gf = \{f\}$

§2 Structure of roots in field

Remainder thm If $f(x) \in k[x]$ and $\alpha \in k$, then you can write
$$f(x) = (x-\alpha)g(x) + c$$
for $g(x) \in k[x]$ and $c \in k$, where $\deg g < \deg f$.

Rmks α root of $f \iff c = 0$
$\iff (x-\alpha) \mid f$

Cor: If $f(x) \in k[x]$ has degree n, # roots of $f(x) \leq n$

Cor If # roots of $f(x) = n$, then $f(x) = a_0(x-\alpha_1)\cdots(x-\alpha_n)$ for roots $\alpha_1, \dots, \alpha_n$.

Fund thm of algebra
If $f(x) \in \mathbb{C}[x]$ is non-constant, then $f(x)$ has a complex root.

Key observation Suppose
$$f(x) = x^n + \cdots + a_1 x + a_0$$
$$= (x-\alpha_1)\cdots(x-\alpha_n)$$
has n roots.

$\rightarrow$ Gives formulas for coefficients in terms of roots

$a_0 = (-1)^n \alpha_1 \cdots \alpha_n$

$a_1 =$

$a_i = (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq n} \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_n}$

$a_{n-1} = -(\alpha_1 + \cdots \alpha_n)$

These are symmetric polynomials

Define elementary sym. polynomials

$$
\left.
\begin{aligned}
s_n &= \alpha_1 \cdots \alpha_n \\
\vdots &= \\
s_k &= \sum_{1 \le j_1 < \cdots < j_k \le n} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_k} \\
s_1 &= (\alpha_1 + \cdots \alpha_n)
\end{aligned}
\right\}
$$

α_i are variables

$n=2$

$$(x-\alpha_1)(x-\alpha_2) = x^2 - (\alpha_1+\alpha_2)x + \alpha_1\alpha_2 X$$